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Gravity waves on water of variable depth

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This is a study of the propagation of gravity waves over a basin in which the propagation distance is large compared with the scale of the bottom topography, which, in turn, is large compared with the depth. Special emphasis is given to the low-frequency part of the spectrum and to geometries containing a beach (see figure 1) because of their importance in tidal wave phenomena. Both reflexion phenomena and the dispersive character of the propagation are accounted for and the non-linear aspects of the large amplification associated with the beach climbing are also included. However, the analysis of problems in which the waves break is valid only up to the inception of breaking; post-breaking phenomena are not treated.

1. Introduction

When an underwater earthquake initiates a Tsunami which propagates across the ocean and is incident on continental and island land masses, the encroachment of the water varies greatly from place to place and depends strongly on the source location. In order to identify the important characterizing features which determine that encroachment, several questions must be answered. In particular, one needs a theory for the propagation of surface waves which can account for the non-linear effects which are important as the waves enter shallow waters; it must be valid for that range of wavelengths which plays a dominant role in the encroachment; and it must be applicable for very general variations in bottom topography whose size is of the order of the average depth and whose typical lateral scale, α^{-1}, is large compared with that depth.

Finally, the description of the waves to which the theory leads must be uniformly interpretable over propagation distances which are large compared with α^{-1}.

In this paper we develop theories which can provide each of these requirements. In § 2 we treat an idealized problem which allows us to identify the important wavelengths and to introduce the appropriate analysis of the nonlinear effects. In § 3, we consider the particular bottom topography which varies only with the co-ordinate in the propagation direction. In § 4 we generalize the theory of § 3 to topographies varying with both horizontal co-ordinates, and in § 5 we modify § 4 to rectify a difficulty which is best identified at the end of § 4.

It is widely known and recorded that, when frictional mechanisms and surface tension are ignored, surface waves on an incompressible fluid are consistent with a theory in which†

$$\phi'_{x'x'}+\phi'_{yy}+\phi'_{z'z'} = 0 \quad \text{in} \quad R, \tag{1.1}$$

$$\bar{n}.\operatorname{grad}\phi' = 0 \quad \text{on} \quad \Gamma_1, \tag{1.2}$$

$$\phi'_t+\tfrac{1}{2}v^2+g\eta = 0 \quad \text{on} \quad y = \eta(x',z',t), \tag{1.3}$$

and

$$\eta_t = \phi'_y-\eta_{x'}\phi'_{x'}-\eta_{z'}\phi'_{z'} \quad \text{on} \quad y = \eta(x',z',t). \tag{1.4}$$

In these equations, the particle velocity is $\bar{v} = \operatorname{grad}\phi'$; x' and z' are horizontal co-ordinates; y is the vertical co-ordinate; $\bar{n}$ is the unit normal to the surface Γ_1 (see figure 1); g is the acceleration of gravity; η is the displacement of the free surface from its disturbance-free location at $y = 0$; and R is the region occupied

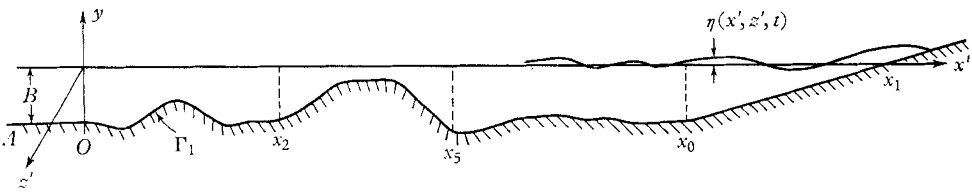

FIGURE 1. Geometry of the general propagation problem.

by the fluid at any time t. It will be convenient to measure the co-ordinates in units of the distance B, shown in figure 1, and, when t is measured in the units $t_0 = (B/g)^{\frac{1}{2}}$, g is replaced in equation (1.3) by the number unity. In such units, the bottom is located at $y = -b(\alpha x',\alpha z') \equiv -b(\beta,\gamma)$ where $\alpha \ll 1$ is so chosen that b_β and b_γ are of order unity. (We have already implied that b is of order unity.) With this notation,‡ equation (1.2) becomes

$$\phi'_y+\alpha b_\beta\phi'_{x'}+\alpha b_\gamma\phi'_{z'} = 0 \quad \text{on} \quad y = -b. \tag{1.5}$$

When η, $\eta_{x'}$ and $\eta_{z'}$ are each small compared with unity, and $v \equiv |\bar{v}|$ is also small compared with unity, the governing equations become linear and, in particular (with $g = 1$), equations (1.3) and (1.4) can be reduced to

$$\phi'_y(x',0,z',t)+\phi'_{tt}(x',0,z',t) = 0, \tag{1.6}$$

where

$$\eta(x',z',t) = \phi'_t(x',0,z',t). \tag{1.7}$$

An alternative approximation which has received some study (Stoker 1948; Carrier & Greenspan 1958), is a 'shallow water theory' which is useful when the lateral scale of the phenomenon is large compared with the depth. The equations of that theory take the form

$$\left.\begin{aligned}[(b+\eta)\,u]_{x'}+[(b+\eta)\,w]_{z'}+\eta_t &= 0,\\ \bar{v}_t+(\bar{v}.\operatorname{grad})\,\bar{v}+\operatorname{grad}\eta &= 0,\end{aligned}\right\} \tag{1.8}$$

and

where u and w are the components of $\bar{v}$ (independent of y) in the x' and z' directions.

† For the analyses in this paper we adopt a 'flat earth' approximation which is inadequate for an *accurate* description of Tsunami propagation over large distances; this limitation does not affect the character of the waves and the techniques of §§ 3, 4, 5 can be readily modified to include such effects. Ordinarily, however, the effect of the curvature of the earth can be approximated adequately by introducing correction factors derived from simple variable-width-channel considerations.

‡ We shall find it necessary to modify this notation only in § 5.

In a typical Tsunami the lateral scale of the important waves is of the order of or larger than the depth, B, and the vertical displacement over most of the trajectory is at most a few feet; accordingly, we shall use the linear theory to describe the propagation over the deep ocean and shall couple the implications of that theory with the non-linear model (equations (1.8)) only in those shallow regions where the wave has been so amplified that such a treatment is necessary.

FIGURE 2. Geometry for the analysis of § 2.

2. A preliminary propagation problem

When a vertical motion of the boundary along AO of figure 2 produces a wave which propagates to the right, the height to which the wave climbs on the sloping 'beach' is much greater than the amplitude of the wave at x_0. However, when the wave amplitude in $0 < x < x_0$ is small compared with the depth, the propagation in that region can be described by the conventional dispersive linear theory in which equations (1.1) and (1.6) are valid and in which equations (1.5) and (1.3) are modified to read

$$\phi_y(x, -b, t) = \text{the imposed boundary motion}, \tag{2.1}$$

and

$$\eta(x,t) = \phi_t(x,0,t). \tag{2.2}$$

Here, we have dropped the primes on ϕ and x (we need that notation only for the more general geometry) and b is the constant, $b = 1$.

In the region, $x_0 < x <$ maximum penetration of wave, the non-linear shallow water theory of Carrier & Greenspan (1958) should suffice for the purposes of this section. We examine its general adequacy more critically, later. In that theory, solutions of equations (1.8) are provided by the formalism whose equations are:

$$(\sigma\psi_\sigma)_\sigma - \sigma\psi_{\lambda\lambda} = 0, \tag{2.3}$$

where

$$u = x\text{-component of velocity} = \psi_\sigma/\sigma\theta^{\frac{1}{2}}, \tag{2.4}$$

$$x - x_1 = \psi_\lambda/4\theta - \sigma^2/16 - u^2/2\theta, \tag{2.5}$$

$$\eta'(\sigma,\lambda) = \tfrac{1}{4}\psi_\lambda - \tfrac{1}{2}u^2, \quad t\theta^{\frac{1}{2}} = \tfrac{1}{2}\lambda - u/\theta^{\frac{1}{2}}. \qquad (2.6), (2.7)$$

In these equations each variable (including the wave amplitude $\eta'(\sigma,\lambda)$) is measured in the same units as in the linear theory.† Any solution of equation (2.3) for which the Jacobian $\partial(x,t)/\partial(\sigma,\lambda)$ is positive in $\sigma > 0$ will have a meaningful interpretation.

The two regions ($x < x_0$ and $x > x_0$) are easily coupled because, whenever the wave amplitude is so small at x_0 that the linear theory is acceptable in $x < x_0$, it is also true that, near x_0 in the non-linear theory,

$$x - x_1 \simeq -\tfrac{1}{16}\sigma^2, \tag{2.8}$$

and

$$t\theta^{\frac{1}{2}} \simeq \tfrac{1}{2}\lambda. \tag{2.9}$$

† Equations (2.4)...(2.8) differ from those listed in Carrier & Greenspan (1958) only because, in that paper, a different set of units was adopted.

Thus, we need only require that the wave proceeding to the right in $x > x_0$ according to equations (2.3)...(2.7) have the same amplitude $\eta(x,t)$ at x_0 as that proceeding to the right in $x < x_0$ according to the linear theory. The same statement holds for the reflected wave proceeding to the left but we shall not want to describe that wave in $x < x_0$ anyway. This matching of η is equivalent to the adoption of a transmission coefficient of unity for waves proceeding through x_0. Except for waves of very great wavelength, the actual transmission coefficient will be very close to unity for slopes θ such that $\theta \ll 1$. The reflexion process will be discussed further in § 3.

Since we shall be interested primarily in the ratio $\eta'_{\max}(0,\lambda)/\eta_{\max}(x_0,t)$ the use of a one-dimensional theory in $x < x_0$ is fully acceptable even when the boundary disturbance does not extend over $-\infty < z' < \infty$. For a given surface motion on an area A' along OA, the wave *magnitude* at x_0 will be exaggerated by the one-dimensional theory but the wave-*form* will be no less realistic than that given by a two-dimensional theory. Furthermore, the amplitude exaggeration can be corrected approximately by reducing the prediction at and beyond $x = x_0$ by the factor $x_0^{\frac{1}{2}}A'^{-\frac{1}{4}}$, where A' is the area over which the wave was initiated.

Consider now the motion in $x < x_0$ which is instigated by the bottom motion

$$\phi_y(x,-1,t) = \begin{cases} 0 & \text{in} \quad t < 0, \\ 0 & \text{in} \quad x > 0, \\ a\, e^{ax} c^2 t\, e^{-ct} & \text{in} \quad x < 0,\, t > 0. \end{cases} \tag{2.10}$$

The exponential forms are chosen purely for convenience. Note particularly that

$$\int_0^\infty dt \int_{-\infty}^0 a\, e^{ax} c^2 t\, e^{-ct}\, dx = 1, \tag{2.11}$$

and

$$\phi_y(x,-1,0) = 0. \tag{2.12}$$

These are useful in what follows, especially in the limit $a \to \infty$, $c \to \infty$. We consider the effect of other initiating ground motions in § 3.

Using the Laplace transform in t and the Fourier transform† in x (as though no change in geometry occurred at x_0), equations (1.1), (1.6) and (2.2) become

$$\overline{\phi}_{yy} - \xi^2 \overline{\phi} = 0, \tag{2.13}$$

$$\overline{\phi}_y(\xi,0,s) + s^2 \overline{\phi}(\xi,0,s) = 0, \tag{2.14}$$

$$\overline{\phi}_y(\xi,-1,s) = \frac{ac^2}{(a-i\xi)(c+s)^2}, \tag{2.15}$$

and the surface displacement is given by

$$\overline{\eta}(\xi,s) = s\overline{\phi}(\xi,0,s). \tag{2.16}$$

The particular form used in equation (2.10) has now accomplished its purpose (no non-homogeneous terms in equations (2.14) and (2.16)) and it now simplifies

† Definition: $\overline{\phi}(\xi,y,s) = \int_{-\infty}^{\infty} dx \int_0^\infty e^{-st} e^{-i\xi x} \phi(x,y,t)\, dt.$

things if we confine our attention to the case $c \to \infty$, $a \to \infty$ so that equation (2.15) becomes

$$\bar{\phi}_y(\xi, -1, s) = 1. \tag{2.15a}$$

The corresponding solution of equation (2.13) is

$$\bar{\phi} = \frac{-(\xi \cosh \xi y - s^2 \sinh \xi y)}{\xi(s^2 \cosh \xi + \xi \sinh \xi)}, \tag{2.17}$$

and, according to equation (2.16),

$$\bar{\eta}(\xi, s) = \frac{-s}{s^2 \cosh \xi + \xi \sinh \xi}. \tag{2.18}$$

Inversion of the Laplace transform over s gives

$$\eta^*(\xi, t) = \cos[tf(\xi)] \operatorname{sech} \xi, \tag{2.19}$$

where η^* is the Fourier transform with regard to x of $\eta(x, t)$, so that

$$\eta(x_0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta^*(\xi, t)\, e^{i\xi x_0}\, d\xi, \tag{2.20}$$

and where $f(\xi) = (\xi \tanh \xi)^{\frac{1}{2}}$.

We now turn to the sloping beach theory and note that, at x_0, according to equation (2.8)

$$\sigma(x_0) \simeq 4(x_1 - x_0)^{\frac{1}{2}} = 4\theta^{-\frac{1}{2}} \equiv L. \tag{2.21}$$

We also have from (2.20) and (2.6) that, for the sloping beach theory, the wave height at x_0 is

$$\begin{aligned} \eta(x_0, t) \equiv \eta'(L, \lambda) = \tfrac{1}{4}\psi_\lambda(L, \lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos(\lambda f(\xi)/2\theta^{\frac{1}{2}}}{\cosh \xi}\, e^{i\xi x_0}\, d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \xi x_0}{\cosh \xi} \exp\{-i\lambda f(\xi)/2\theta^{\frac{1}{2}}\}\, d\xi, \end{aligned} \tag{2.22}$$

and the solution of (2.3) which is bounded at the origin ($\sigma = 0$) and obeys the boundary condition at x_0 is

$$\tfrac{1}{4}\psi_\lambda(\sigma, \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \xi x_0}{\cosh \xi} \frac{\exp\{-i\lambda f(\xi)/2\theta^{\frac{1}{2}}\}\, 2J_0[(\sigma f(\xi)/2\theta^{\frac{1}{2}})]}{H_0^{(2)}[Lf(\xi)/2\theta^{\frac{1}{2}}]}\, d\xi. \tag{2.23}$$

For all but small ξ (in fact, whenever $Lf(\xi)/2\theta^{\frac{1}{2}} > 3$) the Hankel function can be replaced by its asymptotic approximation and we write

$$\tfrac{1}{4}\psi_\lambda(0, \lambda) \simeq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \xi x_0}{\cosh \xi} [\pi L f(\xi)/\theta^{\frac{1}{2}}]^{\frac{1}{2}} \exp\left\{-i \frac{\lambda - L}{2\theta^{\frac{1}{2}}} f(\xi)\right\} d\xi, \tag{2.24}$$

where it is understood that the integral is to be evaluated by the method of stationary phase, where we regard x_0, t and L as large parameters, and where it is understood that the answer is inaccurate for any values of t for which the saddle points are so close to the origin that the Hankel function is not well approximated by this procedure. The result given by the stationary phase calculation is

$$\eta'(0, \lambda) \sim \operatorname{Re}\left[\frac{2f(z)}{\theta x_0}\right]^{\frac{1}{2}} \left[\frac{f'(z)}{-f''(z)}\right]^{\frac{1}{2}} \frac{\exp[i\{x_0[z - f(z)/f'(z)] + \pi/4\}]}{\cosh z}, \tag{2.25}$$

where the saddle points $\pm z$ are the roots of

$$f'(z) = 2x_0\theta^{\frac{1}{2}}/(\lambda - L).$$

This method will give a good approximation to the maximum penetration whenever the value of ξ for which $x_0[\xi - f(\xi)/f'(\xi)] = -\frac{1}{4}\pi$ is such that our approximation to the Hankel function is a good one. That is, equation (2.25) will estimate the penetration reliably for all configurations in which $\theta < (x_0)^{-\frac{1}{3}}$. For such configurations, the penetration is given by

$$\eta'_{\max} = [2/x_0\theta]^{\frac{1}{2}}.$$

The wave height at x_0 as described by equation (2.22) has the uniform asymptotic description (see Appendix):

$$\eta \sim \left[\frac{f(Z) - (Z)f'(Z)}{-3f''(Z)}\right]^{\frac{1}{2}} J[x_0\{f(Z)/f'(Z) - Z\}], \tag{2.26}$$

where $J(\mu) = J_{\frac{1}{3}}(\mu) + J_{-\frac{1}{3}}(\mu)$, and where Z is the positive root of $f'(Z) = x_0/t$.

For values of t such that $Z \ll 1$, equation (2.26) can also be written

$$\eta \sim \frac{(3)^{-\frac{2}{3}}}{[-x_0 f'''(0)]^{\frac{1}{3}}} \frac{F^{\frac{1}{3}} J(F)}{2}, \tag{2.26a}$$

where $$F = x_0\{f(Z)/f'(Z) - Z\}.$$

The maximum value of η as given by equation (2.26a) is about $0{\cdot}33x_0^{-\frac{1}{3}}$ and the amplification provided by the sloping shelf is

$$\frac{\eta'_{\max}(0, \lambda)}{\eta_{\max}(x_0, t)} \sim \frac{(2/x_0\theta)^{\frac{1}{2}}}{0{\cdot}33x_0^{-\frac{1}{3}}} = 4{\cdot}2\theta^{-\frac{1}{2}}x_0^{-\frac{1}{6}}.$$

This ratio is relevant when the initiation of the wave is provided by an upward motion of the bottom surface; when a downward motion produces the wave the relevant amplification factor is the ratio of the negative maxima at $\sigma = 0$ and at $x = x_0$. For that situation,

$$\frac{[-\eta'(0, \lambda)]_{\max}}{[-\eta(x_0, t)]_{\max}} \sim 5{\cdot}6\theta^{-\frac{1}{2}}x_0^{-\frac{1}{6}}.$$

Clearly, some of the amplification associated with large Tsunami run-up can be accounted for without regard to the effects of bottom topography in $x < x_0$ and without regard to dissipative mechanisms.

To calculate the details of the wave-form, one should choose Z (or z) as an independent parametric variable and use the equation defining Z (or z) to find the corresponding value of t (or λ). Equations (2.25) and (2.26) can then be invoked to complete the parametric description of the wave height as a function of t. Without doing any of this, however, we can see that the lateral scale of the head wave is identified by the position of the saddle point corresponding to the maximum wave height. At x_0 this says that wave-numbers of order $x_0^{-\frac{1}{3}}$ are most important. Thus, the lateral scale is several depths and the use of shallow water theory in $x > x_0$ provides a very reasonable approximation. In subsequent sections, however, we shall see that the coupling of the linear and non-linear theories can be accomplished at a shallow part of the shelf; thus, we need no further error estimate for this preliminary study.

In the remaining sections of this paper it will not be possible to use, in a direct way, the Fourier transform with regard to x. Instead, we shall study the propagation of waves of the form† $\chi'(x', y, \alpha, \omega)\, e^{-i\omega t}$ and synthesize over ω when we have found χ' so that ϕ' is described by

$$\phi' = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \chi'(x', y, \alpha, \omega)\, e^{-i\omega t}\, d\omega. \tag{2.27}$$

To do this we must have a specification of $\chi'(0, y, \alpha, \omega)$. To obtain this initial condition for the wave whose initiation is described by equation (2.10), we invert equation (2.17) over ξ; that is, we write (with $s = -i\omega$)

$$\chi'(0, y, \alpha, \omega) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\phi}\, e^{i\xi x'}\, d\xi\right)_{x'=0}. \tag{2.28}$$

The path of integration must pass below $\xi_0(\omega)$, the positive real root of

$$\xi \sinh \xi - \omega^2 \cosh \xi = 0, \tag{2.29}$$

and above the point $-\xi_0(\omega)$ so that equation (2.27) will describe waves which are propagating away from the origin. The residues from the other poles of $\bar{\phi}$ produce contributions to χ' which decay exponentially in x' and we ignore them. Thus, we obtain

$$G = \frac{-2i \cosh[\xi_0(y+1)] \cos \xi_0 x'}{\xi_0 + \sinh \xi_0 \cosh \xi_0}, \tag{2.30}$$

where G is the integral of (2.28) for any x'. In particular, the contribution at $x' = 0$ of the wave proceeding to the right in equation (2.30) is

$$\chi'(0, y, \alpha, \omega) = \frac{-i \cosh[\xi_0(y+1)]}{\xi_0 + \sinh \xi_0 \cosh \xi_0}. \tag{2.31}$$

If one calculates η from equation (2.30) by inverting over ω and using equation (2.16), one recovers the description of η implied by equation (2.20). Thus, the neglect of the non-propagating modes (i.e. the ignored residues of equation (2.28)) in obtaining equations (2.30) and (2.31) is fully justified; we shall use equation (2.31) as the boundary condition on χ' at $x' = 0$ throughout the following analysis.

An alternative study of the waves which prevail in $x < x_0$ can be found in Kajiura (1963).

3. One-dimensional bottom topography

Figure 1 depicts the geometry appropriate to this section. The topography, b, is a function of β only, and, in accord with the remarks in § 2, we want the solution of equations (1.1), (1.5) and (1.6) for which

$$\phi' = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi'(x', y, \alpha, \omega)\, e^{-i\omega t}\, d\omega, \tag{3.1}$$

and

$$\chi'(0, y, \alpha, \omega) = \frac{-i \cosh[(y+1)\xi_0(\omega)]}{\xi_0 + \sinh \xi_0 \cosh \xi_0}. \tag{3.2}$$

† The arguments of χ' will differ in different sections according to the demands of the topography.

In order to obtain a reasonably simple description of such waves which is readily interpreted for large x', we must take advantage of the fact that $\alpha \ll 1$ and of the two lateral scales which will arise, one associated with the wavelength of the wave and one associated with the topography. Accordingly, we anticipate (see Mahony 1962) that it will be useful to define a new potential, χ, in the form

$$\chi' = \chi(x, y, \beta, \alpha, \omega), \tag{3.3}$$

where x is a co-ordinate defined by

$$x = \int_0^{x'} k(\omega, \alpha x'')\, dx''. \tag{3.4}$$

We shall deduce later what k must be.

When we substitute equations (3.1), (3.3) and (3.4) into equations (1.1), (1.5), (1.6) and (3.2), we obtain

$$\chi_{yy} + k^2\chi_{xx} + \alpha k_\beta \chi_x + 2k\alpha\chi_{x\beta} + \alpha^2\chi_{\beta\beta} = 0, \tag{3.5}$$

with

$$\chi_y = \omega^2\chi \quad \text{on} \quad y = 0, \tag{3.6}$$

$$\chi_y = \alpha b_\beta(k\chi_x + \alpha\chi_\beta) \quad \text{on} \quad y = -b, \tag{3.7}$$

and

$$\chi(0, y, 0, \alpha, \omega) = \frac{-i\cosh[(y+1)\,\xi_0(\omega)]}{\xi_0 + \sinh\xi_0\cosh\xi_0}. \tag{3.8}$$

We seek that function χ which obeys equations (3.5) to (3.8) as though β bore no relation to x'; when χ has been found, the solution of the original problem is given by $\chi(x, y, \alpha x', \alpha, \omega)$.

The most convenient description of χ is given by the perturbation series

$$\chi = \chi^{(0)}(x, y, \beta, \omega) + \alpha\chi^{(1)}(x, y, \beta, \omega) + \dots. \tag{3.9}$$

When we defined x in equation (3.4) we anticipated that k should be a function only of ω and β so that, when equation (3.9) is substituted into equation (3.5), we obtain

$$\chi^{(0)}_{yy} + k^2\chi^{(0)}_{xx} = 0, \tag{3.10}$$

$$\chi^{(1)}_{yy} + k^2\chi^{(1)}_{xx} = -k_\beta\chi^{(0)}_x + 2k\chi^{(0)}_{x\beta}. \tag{3.11}$$

Equation (3.6) becomes, for all n,

$$\chi^{(n)}_y = \omega^2\chi^{(n)} \quad \text{on} \quad y = 0, \tag{3.12}$$

and equation (3.7) becomes

$$\left.\begin{aligned} &\chi^{(0)}_y = 0 \\ &\chi^{(1)}_y = -b_\beta\chi^{(0)}_x \\ &\dots\dots\dots\dots \end{aligned}\right\} \text{on} \quad y = -b(\beta). \tag{3.13}$$

Thus,

$$\chi^{(0)} = A^{(0)}(\beta)\cosh[\kappa k(y+b)]\,e^{i\kappa x}, \tag{3.14}$$

and (because of equation (3.12)), κk is taken as the real positive (negative) root of

$$\kappa k \tanh(\kappa k b) = \omega^2, \tag{3.15}$$

in order that, when ω is positive (negative), equation (3.14) will describe a wave proceeding to the right.

We also note that $A^0(0)$ is determined by the initial condition at $x' = 0$. Using the wave initiation mechanism of § 2, we have (from equation (3.8))

$$A^{(0)}(0) = -i/(\xi_0 + \sinh \xi_0 \cosh \xi_0). \tag{3.16}$$

To find $A^{(0)}(\beta)$ and to choose $k(\beta, \omega)$, we must study the equations which determine $\chi^{(1)}$. Equations (3.11), (3.12) and (3.13) imply that

$$\chi^{(1)}_{yy} + k^2 \chi^{(1)}_{xx} = -k_\beta \chi^{(0)}_x - 2k\chi^{(0)}_{x\beta}, \tag{3.17}$$

$$\chi^{(1)}_y(x, 0, \beta, \omega) = \omega^2 \chi^{(1)}(x, 0, \beta, \omega), \tag{3.18}$$

$$\chi^{(1)}_y(x, -b(\beta)\,\beta, \omega) = -b_\beta k \chi^{(0)}_x. \tag{3.19}$$

In equation (3.17), the term $2k\chi^{(0)}_{x\beta}$ will contain the factor $\kappa_\beta x\, e^{ix}$ and, if κ is not independent of β, the perturbation series for χ will be of the form

$$\chi = e^{ix} \sum_{n=0}^{\infty} a_n(\beta, \ldots)\,(\alpha x)^n.$$

When $\alpha x \gg 1$, this is a useless description and our choice of k must help to avoid such a description. Accordingly, we choose $k(\beta, \omega)$ to be such that $\kappa \equiv 1$; this

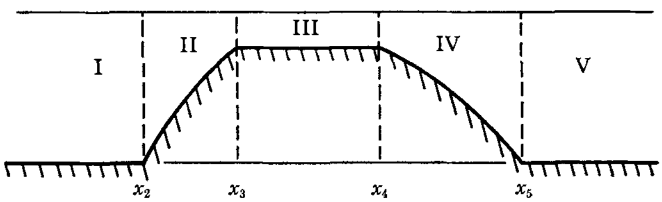

FIGURE 3. An idealized geometry which is useful in calculating the reflexion and transmission properties for the topography in $x_2 < x' < x_5$ of figure 1.

choice and the sentence containing equation (3.15) determine k uniquely. Using this choice, equation (3.17) becomes

$$\begin{aligned} \chi^{(1)}_{yy} + k^2 \chi^{(1)}_{xx} &= e^{ix}\{-ik_\beta A^{(0)} \cosh k(y+b) - 2ik[A^{(0)}_\beta \cosh k(y+b) \\ &\quad + A^{(0)}[k_\beta(y+b) + kb_\beta] \sinh (y+b)\,k]\} \\ &= -ik\, e^{ix} f(y+b), \quad \text{say.} \end{aligned} \tag{3.20}$$

Equation (3.19) becomes

$$\chi^{(1)}_y(x, -b(\beta), \beta, \omega) = -ikb_\beta A^{(0)} e^{ix}. \tag{3.21}$$

To avoid secular terms (i.e. terms like $x^n e^{ix}$), we insist that

$$\chi^{(1)} = F(y, \beta, \omega)\, e^{ix}. \tag{3.22}$$

The most general solution of (3.20) which has this form and which obeys (3.18) is

$$F = -i \int_0^y \sinh [k(y-y')] f(y'+b)\, dy' + A^{(1)}(\beta) \cosh [k(y+b)]; \tag{3.23}$$

this function F also obeys (3.21) only if

$$\int_0^b \left[\left\{ \frac{k_\beta}{k} A^{(0)} + 2A^{(0)}_\beta \right\} \cosh^2 kp + A^{(0)}(k_\beta p + kb_\beta) \sinh 2kp \right] dp = -b_\beta A^{(0)}. \tag{3.24}$$

That is,

$$A^{(0)}(\beta) = A^{(0)}(0)\,[2k(0,\omega)+\sinh 2k(0,\omega)]/[2k(\beta,\omega)\,b(\beta)+\sinh 2k(\beta,\omega)\,b(\beta)]^{\frac{1}{2}}. \quad (3.25)$$

Thus, using $\chi^{(0)}$ as a uniformly valid approximation to χ (and hence to χ'), equations (2.27), (3.14), (3.16) and (3.25) define ϕ' without ambiguity. In particular, the surface displacement is given (using equation (1.7)) by

$$\eta(x',t) \sim \frac{i}{2\pi}\int_{-\infty}^{\infty} \omega A^{(0)}(\alpha x',\omega)\cosh(kb)\exp\left\{i\int_0^{x'} k(\alpha x'',\omega)\,dx'' - i\omega t\right\}d\omega. \quad (3.26)$$

Again, the integral of equation (3.26) is most readily evaluated by a method of stationary phase. We define, for brevity,

$$D(x',\omega) = \int_0^{x'} k(\alpha x'',\omega)\,dx'', \quad (3.27)$$

and we note that, for each large x' and t, the integrand of equation (3.26), contributes effectively to the integral only near the saddle points ω which are the roots of

$$D_{\omega}(x',\omega) - t = 0. \quad (3.28)$$

The symmetry in the definition of k implies that, for each x' and t, there are two such saddle points ($\omega_1 = -\omega_2$). Accordingly, the development in the Appendix is again appropriate and η is given by

$$\eta \sim \frac{2}{\sqrt{3}}\,\omega_0\,A^{(0)}(\alpha x',\omega_0)\cosh[bk(\alpha x',\omega_0)]\left(\frac{-N}{N''}\right)^{\frac{1}{2}} J(-N), \quad (3.29)$$

where $$N = D(x',\omega_0) - t\omega_0, \quad N'' = D_{\omega\omega}(x',\omega_0),$$

and where ω_0 is the positive root of equation (3.28). For $\omega_0 \ll 1$, equation (3.29) can also be written

$$\eta \sim \frac{2(3)^{-\frac{2}{3}}}{[D_{\omega\omega\omega}(\omega=0)]^{\frac{1}{3}}}\,\omega_0\,A^{(0)}(\alpha x',\omega_0)\,(-N)^{\frac{1}{3}} J(-N). \quad (3.30)$$

Despite the tedious calculations which would be needed to find the wave-form η for most values of x' and t, a very simple recipe arises from equation (3.30) when the head wave is near $x' = x_0$. As we saw in § 2, the saddle points corresponding to the head wave description lie very close to the origin (i.e. $\omega_0 \ll 1$) and equation (3.30) is very well approximated by

$$\eta \sim 3^{-\frac{2}{3}} H^{\frac{1}{3}} J(H)/X_2^{\frac{1}{3}}\, 2b^{\frac{1}{4}}, \quad (3.31)$$

where $$X_1 = \int_0^{x'} b^{-\frac{1}{2}}(\alpha x'')\,dx'', \quad X_2 = \int_0^{x'} b^{\frac{1}{2}}(\alpha x'')\,dx'',$$

$$-H = \omega_0 X_1 + \tfrac{1}{6}\omega_0^3 X_2 - \omega_0 t$$

and ω_0 (the positive saddle point) is given by

$$\omega_0^2 = 2(t - X_1)/X_2.$$

The largest positive and largest negative values of η are

$$\eta_{\max} \sim 0{\cdot}33(X_2)^{-\frac{1}{3}} \quad \text{and} \quad (-\eta)_{\max} \sim 0{\cdot}25(X_2)^{-\frac{1}{3}}.$$

The implications of equation (3.31) are these:

When a wave propagates over a basin for which the bottom lies at $b(\alpha x)$, and for which, in particular, $b(0) = b_0$, $b(\alpha x_0) = b_1$, the amplitude ratio $\eta(x_0)/\eta(0)$ is

$$\eta(x_0)/\eta(0) = (b_0/b_1)^{\frac{1}{4}} \left(\int_0^{x_0} b^{\frac{1}{2}}(\alpha x'')\, dx''\right)^{-\frac{1}{3}}.$$

The first factor, which is frequently called Green's formula, is independent of the topographical details and depends only on the initial and final depths; the second factor arises from the dispersion of the wave and cannot differ greatly from $(x_0)^{-\frac{1}{3}}$ even when the depth undulates significantly along the trajectory.

The reflexion process. The foregoing technique does not provide for a description of the effects of reflexions nor, in § 2, was the reflexion at x_0 taken into account. However, Kajiura (1961) has discussed the manner in which such reflexions can be calculated. He notes, in particular, that the reflexion coefficient associated with a given depth transition is very small except in that part of the spectrum for which the wavelength is very large compared with the width of the transition zone; even in that part of the spectrum the reflexion is small unless the depth ratio associated with the transition differs greatly from unity. He also notes that the transmission coefficient differs significantly from unity in a way which is even less sensitive to changes of depth.

Accordingly, we could expect in studying the configuration of figure 1, for example, that reflexions would be important only in $x_2 \leqslant x' \leqslant x_5$ and at $x' = x_0$ (the reflexion in $x' > x_0$ is included in the sloping beach analysis of § 2). These reflexion processes can be studied individually as follows.

When the incident wave at x_0 has the spectrum†

$$\chi_i = A(\omega) \exp\{ix(x', \omega) - ix(x_0, \omega)\},$$

only the long waves will be affected by reflexion and, near x_0, χ_i can be approximated by

$$\chi_i \simeq A(\omega) \exp\{i\omega(x' - x_0)\}.$$

The spectrum of the wave system, including the reflexions, is

$$\chi_1 \simeq A(\omega) \exp\{i\omega(x'-x_0)\} + B(\omega) \exp\{-i\omega(x'-x_0)\} \quad \text{in} \quad x' < x_0.$$

In $x' > x_0$ (but not too close to $x' = x_1$)

$$\psi_1 = C(\omega)\, J_0[2\omega(x_1 - x')^{\frac{1}{2}}/\theta^{\frac{1}{2}}]$$

and, when we require that η and η_x be continuous at $x' = x_0$, we find that

$$C(\omega)/A(\omega) = [J_0(2\omega/\theta) + iJ_0'(2\omega/\theta)]^{-1}.$$

Thus, for example, when reflexion at x_0 is taken into account in the problem of § 2, equation (2.23) is replaced by

$$\tfrac{1}{4}\psi_\lambda(\sigma, \lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos \xi x_0}{\cosh \xi} \frac{\exp\{-i\lambda f(\xi)/2\theta^{\frac{1}{2}}\}\, 2J_0[\sigma f(\xi)/2\theta^{\frac{1}{2}}]\, d\xi}{J_0(Lf(\xi)/2\theta^{\frac{1}{2}}) + iJ_0'(Lf(\xi)/2\theta^{\frac{1}{2}})}.$$

† By a spectrum, χ, we mean, merely, $\phi' = \int_{-\infty}^{\infty} \chi\, e^{-i\omega t}\, d\omega$.

Since $J_0(z)+iJ_0'(z)$ is asymptotically equivalent to $H_0^{(2)}(z)$ (and approximates it well whenever $z > 4$) we see that the reflexion at x_0 is of interest only for very long waves. We also note that the asymptotic recipe given by equation (2.25) overestimates the contribution of the very small ω part of the spectrum; this may be one of several reasons that, in many Tsunami observations, the earliest wave does not have the greatest encroachment.

To describe the implications of the presence of the ridge in $x_2 < x' < x_5$ (see figure 1), one can use the same kind of analysis. It will suffice to approximate the configuration by figure 3 and to apply in each of the regions I, ..., V, the shallow water approximation just used. The boundary conditions at each of $x_2, x_3, \ldots,$ should require continuity of η and $\eta_{x'}$; the result will give an explicit relation of the form

$$\chi_i(\omega, x_5) = F(\omega, \ldots)\,\chi_i(\omega, x_2), \tag{3.32}$$

where the subscript i denotes that part of the wave which is moving to the right (i.e. F is the transmission coefficient of the transition). This equation can be used as a 'boundary condition' to join the solution in $x' < x_2$ to that in $x' > x_5$.

The procedure and results are given for certain geometries by Kajiura and we will not carry out the details here. We note again, however, that the ridge in $x_2 < x' < x_5$ in figure 1 will transmit toward x_0 a wave from which much of the low-frequency content has been depleted.

Alternative initial motions. The initiating ground motion of § 2 is not particularly realistic, but the effect of other motions is easily taken into account. Suppose that, as in the Alaskan earthquake of 1964, the ground motion which initiates the water wave has no net vertical displacement. Such a motion can be modelled by

$$\phi_y(x', -1, t) = \begin{Bmatrix} 0, & t < 0, \\ Ax\, e^{-R|x|}T(t), & t > 0, \end{Bmatrix} \tag{3.33}$$

instead of equation (2.10). This initial condition has the transform

$$\bar{\phi}_y(\xi, -1, s) = \frac{4iRA\xi}{(\xi^2+R^2)^2}\,\bar{T}(s) \equiv \bar{X}(\xi)\,\bar{T}(s), \tag{3.34}$$

and, when this is used instead of equation (2.15a), the function $\bar{\phi}$ given by equation (2.17) is modified only in that the right side of equation (2.17) is multiplied by the right side of equation (3.34). Thus the description of $\eta(x,t)$ given in § 2 is 'modulated' by the temporal spectrum of $T(t)$ and the spatial spectrum of $X(x)$. The small low-frequency (long wavelength) content implied by the form of $X(\xi)$ and the low-frequency content of $\bar{T}(s)$ could both contribute in an important way to the observation mentioned earlier that the first wave is not always that with the greatest encroachment.

4. Two-dimensional bottom topography

We turn now to a study of waves which are initiated as in §§ 2 and 3 but which propagate over a bottom described by

$$y = b(\alpha x', \alpha z') = b(\beta, \gamma).$$

If we anticipate for the moment that the propagation will occur primarily in the x' direction, we must also anticipate that the wave can be described by equations (3.1) and (3.3) except that χ' and χ will depend on z' *but only through a dependence on* γ. Thus, in equation (1.5) (the only equation through which the bottom topography enters the analysis of § 3) the final term $\alpha b_\gamma \phi'_{z'}$ can be written $\alpha^2 b_\gamma \phi'_\gamma$ and nowhere in the analysis of § 3 will any modification of the procedure be implied. Thus equations (2.27), (3.14), (3.16), (3.25) and (3.26) provide a zero-order description of the wave propagation.

However, there *is* a modification in the interpretation and in the uniformity of that description! It is implicit in equation (3.15) (where we still insist that $\kappa = 1$) that, since $b = b(\beta, \gamma)$, k depends on γ (and hence z') as well as on β. This implies, in turn, that x as defined by equation (3.4) is now a function of z' as well as of x'.

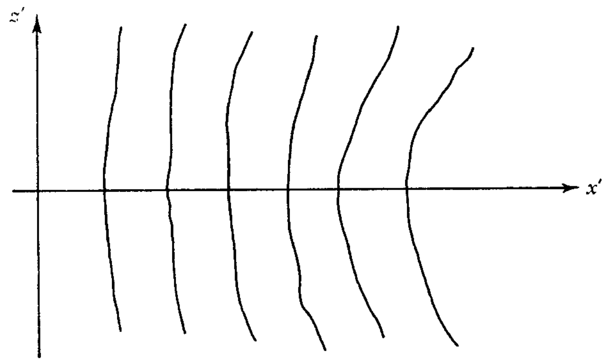

FIGURE 4. Successive positions of a wave crest proceeding to the right for a topography such that
$$\int_0^{x'} b(\alpha x'', \gamma_1)\, dx'' > \int_0^{x'} b(\alpha x'', \gamma_2)\, dx'' \quad \text{when} \quad |\gamma_1| > |\gamma_2|.$$
The irregularities in the crest loci are associated with the topographical irregularities whereas the systematic change in the crest curvature is associated with the systematic depth dependence on z'.

In view of this, the exponent in equation (3.26) is also a function of z' and equation (3.28) becomes
$$D_\omega(x', \gamma, \omega) - t = 0. \tag{4.1}$$

This dependence on γ implies that, at a given time t, the distance x' to which a particular wave crest (or other characterizing feature) has advanced will differ for different values of z'. Figure 4 indicates the successive positions of a given crest for a topography in which $\int_0^{x'} b(\alpha x'', \gamma)\, dx''$ is an increasing function of $|\gamma|$ for all x'.

We call the time-dependent position of any particular wave crest $X(z', t)$ and we note that, for topographies such as that implied by figure 4, the theory we have used is entirely adequate if and only if $\partial X/\partial z' \ll 1$ everywhere in the domain of interest.

Conversely, if $\partial X/\partial z'$ becomes of order unity in the region of interest, it is clear that the theory is inadequate and that the description afforded by equation (3.26) is not a uniformly valid approximation.

We can see how the non-uniformity enters the mathematical analysis by studying the equations governing $\chi^{(2)}(x, y, \beta, \gamma, \omega)$. The difficulty arises through equation (1.5), whose second-order contribution (in terms of $\chi^{(0)}$ and $\chi^{(1)}$) is

$$\chi_y^{(2)} = -b_\beta(k\chi_x^{(1)} + \chi_\beta^{(0)}) - b_\gamma \chi_\gamma - b_\gamma \chi_x^{(0)} \left(\int_0^{x'} k_\gamma(\alpha x'', \gamma, \omega)\, dx''\right). \tag{4.2}$$

The final parenthesis of equation (4.2) will increase monotonically with x' for the topography which led to figure 4 (and for many other topographies) and, thus, the final term of equation (4.2) is reminiscent of the $x'\, e^{ikx'}$ which led to the procedure initiated on page 11 with equation (3.3) *et seq.*

To avoid the non-uniformity implicit in equation (4.2), we must make a major alteration in the procedure. Before doing so, however, we must emphasize again that the alternative procedure of the forthcoming section is neither necessary nor desirable in any investigation in which, over the domain of interest, the crests (or the η-constant lines, *etc.*) are so located that

$$\partial X(z', t)/\partial z' \ll 1.$$

5. An alternative treatment for two-dimensional topographies

Suppose now that the bottom topography is described by

$$y = -b(\beta, \gamma, \epsilon z'), \tag{5.1}$$

where any systematic depth dependence on z' is identified with the dependence of b on $\epsilon z'$ and where the dependence on γ is identified with the irregular aspect of the topography. A reasonably unpleasant illustrative example of such a description is given by

$$b = 1 + \tfrac{1}{2}\sin\beta \sin\gamma - \tfrac{1}{4}(1 + \tfrac{1}{4}\sin^2\beta)\, e^{-(\epsilon z')^2}.$$

With this function, b, the wave would be propagating along a ridge 'centred' on $z' = 0$.

Even when $\epsilon < \alpha$, it is convenient for very large x' to find ϕ' as follows.

We again adopt equations (2.27) and (3.4) but, instead of equations (3.3) and (3.9) we must use

$$\chi' = \chi(x, y, z, \beta, \gamma, \alpha, \omega, \epsilon), \tag{5.2}$$

and

$$\chi = \chi^{(0)}(x, y, z, \beta, \gamma, \omega, \epsilon) + \alpha\chi'(\ldots) + \ldots, \tag{5.3}$$

where $z = \epsilon z'$. We must also expect that k will depend on β, γ and ω but, if the description is to be uniformly valid, k must not depend on z; it was through the dependence of k on the systematic depth variation with z' that we found the troubles which necessitated this alternative treatment.

When we substitute equations (2.27), (3.4) and (5.2) into equations (1.1), (1.2) and (1.6), we obtain

$$\chi_{yy} + \epsilon^2\chi_{zz} + k^2\chi_{xx} + \alpha\left[2k\chi_{\beta x} + k_\beta\chi_x + 2\epsilon\chi_{\gamma z} + 2\epsilon\left(\int_0^{x'} k_\gamma(\alpha x'', \gamma, \omega)\, dx''\right)\chi_{xz}\right]$$
$$+ \alpha^2\left[\chi_{\beta\beta} + \chi_{\gamma\gamma} + 2\left(\int_0^{x'} k_\gamma\, dx''\right)\chi_{x\gamma} + \left(\int k_\gamma\, dx''\right)^2\chi_{xx} + \left(\int k_{\gamma\gamma}\, dx''\right)\chi_x\right] = 0, \tag{5.4}$$

$$\chi_y - \omega^2\chi = 0 \quad \text{on} \quad y = 0, \tag{5.5}$$

and, on $y = -b$,

$$\chi_y + \epsilon^2 b_z \chi_z + \alpha\left[b_\beta k\chi_x + \epsilon b_\gamma \chi_z + \epsilon b_z \chi_x \int_0^{x'} k_\gamma\, dx'' + \epsilon b_z \chi_\gamma\right] + \alpha^2\left[b_\beta \chi_\beta + b_\gamma \chi_\gamma + b_y \chi_x \int_0^{x'} h_\gamma\, dx''\right] = 0. \quad (5.6)$$

We insist again (in order to avoid secular terms) that χ depend on x only through the factor e^{ix} so that equation (5.3) becomes

$$\chi = e^{ix}[W^{(0)}(y, z, \beta, \gamma, \omega, \epsilon) + \alpha W^{(1)}(\ldots) + \alpha^2 \ldots]. \quad (5.7)$$

Using equations (5.4), (5.5), (5.6) and (5.7), we find that $W^{(0)}$ must obey

$$W^{(0)}_{yy} + \epsilon^2 W^{(0)}_{zz} - k^2 W^{(0)} = 0, \quad \text{in} \quad R, \quad (5.8)$$

$$W^{(0)}_y - \omega^2 W^{(0)} = 0, \quad \text{on} \quad y = 0, \quad (5.9)$$

and

$$W^{(0)}_y + \epsilon^2 b_z W^{(0)}_z = 0, \quad \text{on} \quad y = -b(\beta, \gamma, z). \quad (5.10)$$

This is a conventional homogeneous two-dimensional problem which has many eigenvalues† k and eigensolutions $W^{(0)}$. The eigenvalues, k, will depend on β and γ but not directly on z. The solutions can be found when the dependence of b on z is specified; when $\epsilon \ll 1$ (ϵ is certainly at least as small as α for oceanic problems) the eigenfunctions and eigenvalues of equations (5.8), (5.9) and (5.10) can be found easily by using a perturbation process in ϵ.

Each of the eigenfunctions $W^{(0)}$ which has a real eigenvalue corresponds to a propagating 'mode' and the combination of eigenfunctions which must be used is determined when the conditions at $x' = 0$ are invoked. In the early stages of the propagation, many eigenfunctions would be needed to describe the configuration (including, for example, the curved 'wave fronts' illustrated in figure 4) but, at very large x', only the fundamental mode will contribute to the head wave (the other modes travel too slowly). Thus, we confine our attention to the fundamental mode which we will continue to call $W^{(0)}$. Once we establish that equation (5.7) can provide a uniformly valid representation, χ will be seen to be uniformly approximated by

$$\chi = e^{ix} W^{(0)}.$$

We note that, in contrast to the remarks in § 4, there are no terms in equations (5.4), (5.5) and (5.6) of the form $f(x')\, e^{ix}$ where $f(x')$ increases steadily in x'. In particular,

$$\int_0^{x'} k_\gamma(\alpha x'', \beta, \omega)\, dx''$$

and the other integrals in those equations do not 'accumulate' with x'; the price we pay for this is the necessity of solving equations (5.8), (5.9) and (5.10) instead of the problem of § 3.

We now denote $W^{(0)}$ by $A^{(0)}(\beta, \gamma, \omega)\, w^{(0)}$ where $w^{(0)}$ is the fundamental eigensolution of equations (5.8), (5.9) and (5.10) normalized in any convenient way. $A^{(0)}$ must be determined, as in § 3, by the requirement that $\chi^{(0)}$ satisfy the initial

† For many geometries the spectrum will contain both a discrete part and a continuous part; this, in principle, causes no difficulty.

conditions and that $\chi^{(1)}$ should not contain secular terms. To assure the latter requirement, we write

$$W^{(1)}_{yy}+\epsilon^2 W^{(1)}_{zz}-k^2 W^{(1)} = F^*, \tag{5.11}$$

where, as is dictated by equation (5.4),

$$F^* = 2ikW^{(0)}_{\beta}+ik_{\beta}\,W^{(0)}+2\epsilon W^{(0)}_{\gamma z}+2i\epsilon\left(\int_0^{x'} k_{\gamma}(\gamma x'',\gamma,\omega)\,dx''\right)W^{(0)}_{z}, \tag{5.12}$$

and where the integral in equation (5.12) is a function of (β,γ,ω), which, as we noted above, does not grow with x'. For many geometries, the two last terms in F^* are numerically negligible and can be ignored.

We also recall that

$$w^{(0)}_{yy}+\epsilon^2 w^{(0)}_{zz}-k^2 w^{(0)} = \Delta w^{(0)}-k^2 w^{(0)} = 0, \tag{5.13}$$

where Δ denotes the Laplace operator in y and z' (γ is still treated as an independent parameter, of course), and we also have

$$\left.\begin{aligned} w^{(0)}_{y}-\omega^2 w^{(0)} = W^{(1)}_{y}-\omega^2 W^{(1)} = 0 \quad &\text{on} \quad y=0,\\ w^{(0)}_{y}+\epsilon^2 b_z w^{(0)}_{z} = 0 \quad \text{on} \quad &y=-b, \end{aligned}\right\} \tag{5.14}$$

and

$$\left.\begin{aligned} w^{(1)}_{y}+\epsilon^2 b_z w^{(1)}_{z} = &-ib_{\beta}kA^{(0)}w^{(0)}-\epsilon b_{\gamma}A^{(0)}w^{(0)}_{z}-i\epsilon b_z A^{(0)}w^{(0)}\int_0^{x'} k_{\gamma}\,dx''-\epsilon b_z[A^{(0)}w^{(0)}]_{\gamma}\\ &\equiv G, \quad \text{on} \quad y=-b. \end{aligned}\right\} \tag{5.15}$$

We multiply (5.11) by $w^{(0)}$ and (5.13) by $W^{(1)}$ to obtain (for channels of finite width in the domain $z_0 < z' < z_1$, the limits on the z' integration would be z_0 and z_1)

$$\left.\begin{aligned} &\int_{-\infty}^{\infty}\int_{-b}^{0}(w^{(0)}\Delta W^{(1)}-W^{(1)}\Delta w^{(0)})\,dy\,dz'\\ &\quad=\int_{-\infty}^{\infty}\int_{-b}^{0} w^{(0)}F^*\,dy\,dz' = \int_{\Gamma} w^{(0)}(\overline{n}\,.\,\text{grad}\,W^{(1)})\,dS\\ &\quad=\int_{-\infty}^{\infty}(1+b_{z'}^2)^{-\frac{1}{2}}w^{(0)}[W^{(1)}_{y}+\epsilon^2 b_z W^{(1)}_{z}]\,dz' = \int_{-\infty}^{\infty}(1+b_{z'}^2)^{-\frac{1}{2}}w^{(0)}G\,dz'. \end{aligned}\right\} \tag{5.16}$$

In the line integral, S denotes distance along Γ which is the curve $y=-b$ and, in the subsequent integrals, each integrand is evaluated at $y=-b$.

Since G contains both $A^{(0)}_{\gamma}$ and A^0 and since F contains $A^{(0)}_{\beta}$, $A^{(0)}_{\gamma}$ and A^0, we extract from equation (5.16), as a first-order linear partial differential equation for $A^{(0)}$,

$$\int_{-\infty}^{\infty}\int_{-b}^{0} w^{(0)}F^*\,dy\,dz' = \int_{-\infty}^{\infty}(1+b_{z'}^2)^{-\frac{1}{2}}w^{(0)}G\,dz'. \tag{5.17}$$

In this analysis, equation (5.17) plays exactly the role that equation (3.24) played in § 3. No further reduction of equation (5.17) is informative until a particular dependence of b on z' is adopted. Equation (5.17), together with an initial condition at $x'=0$ (equation (3.2) for example), completely specifies χ and only the inversion over ω remains.

6. Discussion

The theory of §§ 3 and 4 can be used in a straightforward manner for any problem in which the topography and the propagation path length do not lead to highly curved crest loci. The domain covered by that theory can include much of the sloping shelf, where interest is usually centred. On the shallower part of the shelf, non-linear contributions to the dynamics become important but only where the water is so shallow that, in the non-linear régime, shallow water theory is a valid approximation. Thus, an easily justifiable coupling of the theories of §§ 3, 4 or 5 and the non-linear theory of § 2 can be made, not at x_0 of figure 1 but at a point x_2 which lies to the left of x_1 but far to the right of x_0. If, at such a matching region, the wave-fronts have a curvature which requires a two-dimensional non-linear shallow water analysis, the formalism of § 2 does not suffice. For such a situation, the underlying ideas of §§ 3–5 should allow an extension of that shallow water theory which would suffice. The same ideas should be applicable when the shelf itself has lateral (z co-ordinate) topographical variations but we will not pursue those extensions of the theory here.

The theory of § 5 can also be used to study the composite effects of waves which propagate across a deep basin and are superimposed with edge waves which arrive from the same source at the same time to produce large run-up. Such interference phenomena *may* be an important feature of the encroachment pattern, especially when the initiating ground motion is in or near a continental shelf.

Although the foregoing techniques seem adequate for the study of the propagation of Tsunamis over the deep ocean and for the study of the run-up which accompanies normal incidence on a sloping beach of waves which do not break, there are many situations for which the foregoing study is entirely inadequate. Among the outstanding questions which remain unanswered and which should not go unmentioned are:

(1) What is the run-up when the wave incidence is oblique?

(2) What is the run-up when the wave breaks and a bore is formed?

(3) How are the foregoing motions coupled to the motion in bays or harbours when the coast-line is not nearly straight but contains major indentations?

(4) How can one interpret the results of model experiments in the context of oceanic phenomena taking account of the fact that dissipation plays a very disproportionate role in the comparatively small-scale experiment?

Until these questions can be answered quantitatively, the scientifically sound design of protective measures will not easily be accomplished.

The author greatly appreciates the hospitality of the University of Western Australia, where this work was performed, and the support of the Guggenheim Foundation and a Fulbright Grant, whose generosity made this visit possible.

Appendix

Although the asymptotic results given in equations (2.26) and (3.29) are implicit in Chester, Friedman & Ursell (1957), it seems worth while to give a simple heuristic derivation of those results and of equation (2.26a). We confine our attention to an integral of the type

$$I = \int_{-\infty}^{\infty} \exp\{i\lambda[h(z) - \alpha z]\}\, dz, \tag{A 1}$$

where $\lambda \gg 1$, $h(-z) = -h(z)$ and where, for $|z| \ll 1$,

$$h(z) = a_3 z^3/3! + a_5 z^5/5! + \ldots,$$

with $a_3, a_5, \ldots = O(1)$. We also require that, when $\alpha > 0$, there are two and only two real values of z such that $h(z) = \alpha$. The reader will see that the procedure is easily generalized to other integrals in which two saddle points coalesce as a parameter α tends to α_0.

When $\alpha > 0$, an asymptotic approximation to I is

$$I \sim \left(\frac{2\pi}{\lambda h''(z_0)}\right)^{\frac{1}{2}} [\exp\{i\lambda(h(z_0) - \alpha z_0) + \tfrac{1}{4}i\pi\} + \exp\{-i\lambda(h(z_0) - \alpha z_0) - \tfrac{1}{4}i\pi\}], \tag{A 2}$$

where z_0 is the positive root of

$$h'(z) = \alpha. \tag{A 3}$$

The first exponential in equation (A 2) arises via the conventional method of stationary phase as the contribution of the saddle point at z_0; the second is provided by the saddle point at $-z_0$. However, as α tends to zero for fixed λ, $f''(z_0)$ also tends to zero and equation (A 2) fails to provide a useful result. It is clear that, when α is small enough, the variation with z of the exponent in the integrand of equation (A 1) is dominated near the saddle points, z_i, by

$$h'''(z_i) \simeq h'''(0)$$

rather than by $h''(z_i)$. Accordingly, one wonders whether the retention of the effects of $h'''(z_i)$ in a simple extension of the usual saddle-point method might not provide a result which is valid for all α in $0 \leqslant \alpha < \infty$.

The behaviour of $h(z) - \alpha z$ is well approximated near $z < z_0$ by

$$h - \alpha z \simeq P(z) \equiv h(z_0) - \alpha z_0 + \tfrac{1}{2}[(z - z_0)^2]\, h''(z_0) + \tfrac{1}{6}[(z - z_0)^3]\, h'''(z_0). \tag{A 4}$$

When, in equation (1), $h(z) - \alpha z$ is replaced by this polynomial approximation, we find that

$$\begin{aligned} L &\equiv \int_{-\infty}^{\infty} \exp\{i\lambda P(z)\}\, dz \\ &= \exp\{i\lambda[h(z_0) - \alpha z_0]\} \frac{\pi}{3^{\frac{1}{6}}} [\lambda f'''(z_0)]^{-\frac{1}{3}} A^{\frac{1}{3}} \exp\{iA\} [\exp\{\tfrac{1}{6}i\pi\} H^{(1)}_{\frac{1}{3}}(A) \\ &\qquad + \exp\{-\tfrac{1}{6}i\pi)\, H^{(2)}_{\frac{1}{3}}(A)], \end{aligned} \tag{A 5}$$

where $$A = \lambda[h''(z_0)]^3/3[h'''(z_0)]^2.$$

Using the asymptotic evaluation of the Hankel functions for large A, we see that the contribution $H^{(2)}_{\frac{1}{3}}(A)$ is provided by the saddle point at $z = z_0$ whereas the contribution $H^{(1)}_{\frac{1}{3}}(A)$ is identified with the other saddle point of $P(z)$. Thus, the contribution to L and to I of the saddle point at z_0 is

$$K = \exp\{i\lambda[h(z_0) - \alpha z_0]\}\frac{\pi}{3^{\frac{1}{6}}}[\lambda f'''(z_0)]^{-\frac{1}{3}}A^{\frac{1}{3}}\exp\{i(A - \tfrac{1}{6}\pi)\}H^{(2)}_{\frac{1}{3}}(A). \tag{A 6}$$

Since $h(z)$ is an odd function of z, we conclude that

$$I \sim 2\,\mathrm{Re}\,K. \tag{A 7}$$

The usual error estimates indicate that

$$I - 2\,\mathrm{Re}\,K = \lambda^{-\frac{2}{3}}O(I).$$

For $z_0 \ll 1$, $h(z_0)$, $h''(z_0)$ and $h'''(z_0)$ can be described by power series in z_0 and equation (A 7) can be written in the two forms

$$I \sim \frac{2\pi}{3^{\frac{1}{3}}}[\lambda h'''(0)]^{-\frac{1}{3}}[Q(z_0)]^{\frac{1}{3}}J[Q(z_0)], \tag{A 8}$$

or
$$I \sim 2\pi 3^{-\frac{1}{2}}[Q(z_0)/-Q''(z_0)]^{\frac{1}{2}}J[Q(z_0)], \tag{A 9}$$

where $\quad Q(z) = -\lambda[h(z) - \alpha z] \quad$ and $\quad J(Q) = J_{\frac{1}{3}}(Q) + J_{-\frac{1}{3}}(Q)$.

Although these formulae were derived only for $z_0 \ll 1$, equation (A 9) is asymptotically equivalent to equation (A 7) for *all* real z_0. Furthermore, the use of the cubic approximation

$$h(z) - \alpha z \simeq a_3 z^3/6 - \alpha z$$

in equation (1) when $\alpha \leqslant O(\lambda^{-\frac{1}{3}})$ leads directly to equation (A 9) and to an error estimate $E = O(\lambda^{-\frac{2}{3}})$.

Thus, the errors in equation (A 9) are of order $\lambda^{-\frac{2}{3}}$ for all $\alpha \geqslant 0$ and we see that equation (A 9) is a uniformly valid approximation to I. We also see that equation (A 8) can be used instead of equation (A 9) when $\alpha \leqslant O(\lambda^{-\frac{1}{3}})$.

Equations (2.26) and (3.29) are obtained when we let $\lambda = t$ and $\alpha = 1 - x_0/t$ and when we note that, for $m(-z) = m(z)$,

$$I^* = \int_{-\infty}^{\infty} m(z)\exp\{i\lambda[h(z) - \alpha z]\}\,dz \sim m(z_0)\,I. \tag{A 10}$$

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